MAXIMIZING ADDITIVITY OF SCALE VALUES

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In research to optimize product acceptability from the standpoint of consumer satisfaction, it is sometimes necessary for the measurements of preference to possess the property of additivity. The additivity is a necessary condition of optimization analysis if preferences for articles are separately measured under different conditions and the summation of these preferences maximized.

The problem of additivity of scale values is illustrated by figure 1. In this diagram OA, OB, OC and OD represent measurements of the degree of consumer desire for stimuli A, B, C and D which we will say are apples, bananas, cherries and dates. In the situation where the consumer is indifferent to whether he has apples and bananas, or has cherries, assuming these articles are independently consumed, measurement of desire for cherries should equal the sum of the measurements of desires for apples and for bananas. That is, OC should equal OA plus OB. In the diagram it is apparent that this relationship does not hold, since OC is smaller in length than the sum of the distances OA plus OB. Similarly, if the consumer is indifferent to whether he has bananas and cherries or has dates, OB plus OC should equal OD.

However, a transformation of the form $S' = \phi(S)$ which shifts the origin and alters the relationships of proportionality among the scale values, represented by the curve, changes the scale values into O'A', O'B', O'C' and O'D' for which the relationship of additivity does hold for the relationships of indifference specified. A scale form for whose values the relationship of additivity holds reflects the natural way in the personality system of the consumer whereby separate desires combine to determine choice.

Correction of Relationships of Proportionality Among Scale Values

L. L. Thurstone and L. V. Jones [2] have published a procedure for achieving additivity which they use to correct scale values for the zero point of the scale. In their procedure they use stretching factors expected from the error variance in comparing single-single, single-double or double-double stimuli. The zero point is then taken so as to minimize errors in applying the relationship of addition to stimuli scaled separately and in doublets.

In the diagram origin O" is located so as to minimize errors in applying the relationships of addition O"A + O"B = O"C and O"B + O"C = O"D when the same interval OO" is subtracted from all of the scale values. It is apparent from the diagram that this is not the full correction which is possible. From an algebraic standpoint it is a correction only for the first term in a series of powers to define new scale values S' from the initial scale values S in order to minimize errors in applying the relationship of addition to stimuli for which desires are equivalent. Beyond correcting for the zero point and using three stretching factors for adjusting deviates from comparing single-single, single-double and double-double stimuli, a functional transformation can be applied to improve additivity of scale values further.

We seek a transformation of scale values S into new scale values S' which maximize their additivity, that is, which minimize the error variance in applying the relationship of additivity. The expression to be minimized is

$$\frac{\sum_{m} [S_{ij}' - (S_i' + S_j')]^2 / m}{\left[\sum_{m+n} S_{ij,i,j}'^2 - \left(\sum_{m+n} S_{ij,i,j}' \right)^2 / (m+n) \right] / (m+n)} \tag{1}$$

The numerator represents the error variance in applying the relationship of addition to the $\underline{m}$ pairs of scale values S_i' and S_j' for single stimuli in order to equal scale values S_{ij}' for double stimuli. This is divided by the scale variance of the $\underline{n}$ scale values for single stimuli and the $\underline{m}$ scale values for double stimuli. The denominator is needed in order to consider error variance in relation to the numerical spread in the scale values to which the relationship of addition is applied. An alternative procedure is to minimize the numerator subject to the constraint that the denominator is constant.

We seek constants in a functional transformation $S' = \phi(S)$ which will minimize expression (1). The scale values to be transformed are the original unweighted paired comparison scale values before applying stretching factors to the deviates. A more highly desired single stimulus with multiple uses is to be given the same mathematical treatment as a multiple stimulus. In either case the size of the discriminal error in making comparisons of stimuli would be similarly affected, and a single functional transformation without use of stretching factors seems sufficient.

Maximizing Additivity of Scale Values From the Thurstone-Jones Data

A linear-cubic function was selected for accomplishing the transformation of initial scale values into ones which are additive, the function having the following form.

$$S' = (S + a) + b\,(S + a)^3 . \tag{2}$$

More than two power terms were not introduced into the analysis since there is question of whether the data are adequate for more detailed treatment. A cubic, rather than a squared, term was used since this produced a symmetric transformation for positive or negative scale values. The way in which the constant $\underline{a}$ is employed gives a mathematically meaningful adjustment for the zero point in the initial scale values distinct from altering the proportional relationships of scale values to each other. An additional constant would be required as coeffi-

TABLE 1

TRANSFORMATIONS OF PAIRED COMPARISON SCALE VALUES FROM THURSTONE-JONES DATA

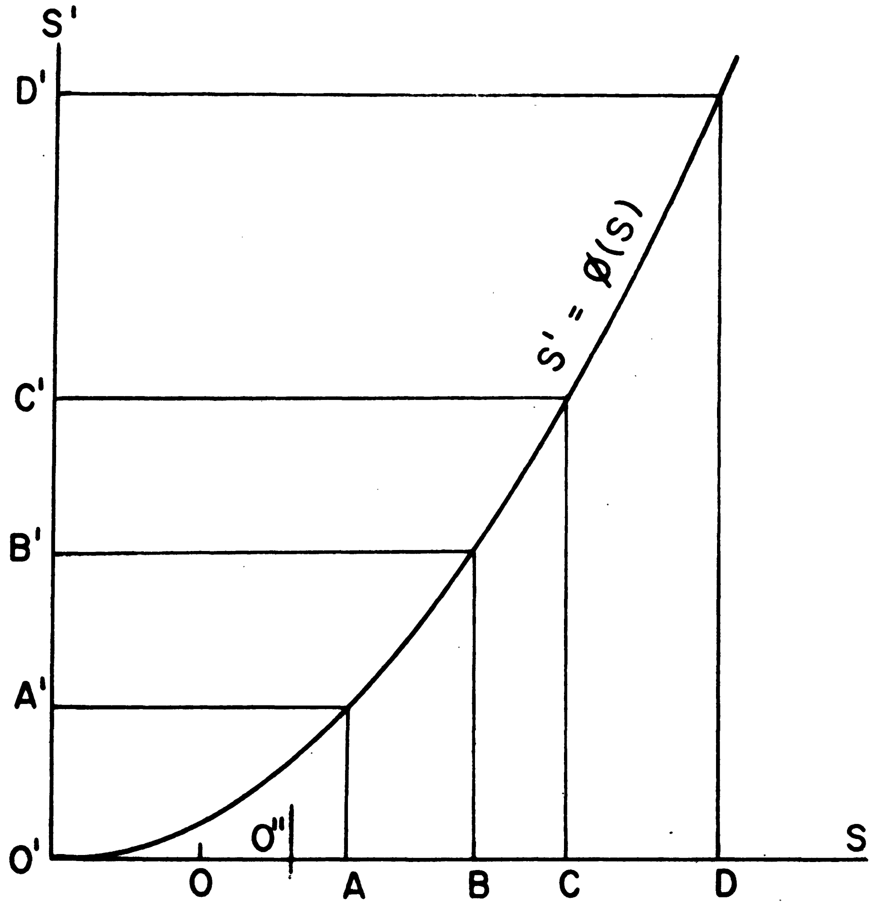
Stimulus	Treatment of Scale Values			
	(I) No corrections applied to scale values	(II) Corrected for zero point, normal deviates unweighted	(III) Corrected by weighting normal deviates and altering zero point	(IV) Scale Values transformed to maximize additivity
A. Briefcase	-1.04	.30	.62	.69
B. Dictionary	-1.06	.28	.61	.66
C. Record Player	.39	1.73	2.81	2.97
D. Desk lamp	- .63	.71	1.26	1.19
E. Pen & pencil set	- .34	1.00	1.68	1.60
A. & B.	- .42	.92	1.48	1.48
A. & C.	.76	2.10	3.72	3.89
A. & D.	- .14	1.20	2.02	1.92
A. & E.	.06	1.40	2.40	2.28
B. & C.	.73	2.07	3.65	3.81
B. & D.	- .27	1.07	1.76	1.71
B. & E.	- .09	1.25	2.12	2.01
C. & D.	.94	2.28	4.07	4.41
C. & E.	.93	2.27	4.10	4.38
D. & E	.18	1.52	2.63	2.52

TABLE 2

PRECISION OF TRANSFORMATIONS OF PAIRED COMPARISON SCALE VALUES

Statistical measure	Treatment of Scale Values			
	(I) No corrections applied to scale values	(II) Corrected for zero point, normal deviates unweighted	(III) Corrected by weighting normal deviates and altering zero point	(IV) Scale Values transformed to maximize additivity
Standard error in adding scale values, $SE[S_{ij} - (S_i + S_j)]$	1.357	.212	.228	.191
Standard error of scale values, $SE(S_{ij,i,j})$	.783	.123	.132	.110
Standard deviation of scale values, $SD(S_{ij,i,j})$	.635	.635	1.120	1.217
Precision of scale values, $SE(S_{ij,i,j})$ / SD	1.23	.19	.12	.09

FIGURE 1



cient for the linear, as well as the cubic, term if the procedure were followed of minimizing the numerator of (1) under the constraint of constant variance in the scale values. This alternative procedure increases somewhat the complexity of the non-linear system of equations to be solved.

Substituting (2) in (1) and minimizing with respect to $\underline{a}$ and $\underline{b}$ gives the two stationary equations whose solution defines $\underline{a}$ and $\underline{b}$.

The original unweighted paired comparison scale values, not given by Thurstone and Jones, were computed from the unweighted normal deviates in their table 2 [2, p. 464]. These scale values, recorded in column I of table 1 of the present paper, were adjusted before furtheranalysis by starting the lowest value at 0 to simplify computation of powers of values. From these adjusted scale values were computed a long series of terms up to the 6th power required for solving the stationery equations. The details of the algebra are not reproduced here. A simpler route of calculation may be feasible than that which was followed here. After substituting these figures in the two partial derivative equations, the solution of the equations was obtained uniquely yielding $\underline{a}$ = .64 and $\underline{b}$ = .096. Since the initial values in column I of the table were increased by 1.06 before computation of the transformation, the value for $\underline{a}$ applied to these values becomes 1.70.

Relative Precision of Scale Forms

Table 1 records several sets of scale values computed from the Thurstone-Jones data by alternative procedures. These scale forms show the effect of successive types of correction. Column I shows paired comparison values computed from unweighted normal deviates. Column II shows these same values together. Column III records scale values computed by Thurstone and Jones after weighting normal deviates with three stretching factors and adjusting for the natural zero point. The normal deviates were weighted by them to allow for the expected increase in the discriminal dispersion in comparing multiple stimuli. In column IV are scale values computed from the unweighted values in column I by the transformation

$$S' = (S + 1.70) + .096(S + 1.70)^3 . \qquad (5)$$

Table 2 gives the standard errors resulting in adding values, the standard errors estimated for the values themselves, the standard deviations of values, and the relative precisions of the sets of scale values in table 1. The error variance in applying the relationship of addition to scale values is equal to the sum of the variances of the errors of the three values in the relationship of addition. The standard error of individual scale values is then obtained by dividing the standard error in adding values by $\sqrt{3}$. The precision of the scale can then be defined as the ratio of the standard error of scale values divided by the standard deviation of the scale values.

As evidenced by their additivity, the precisions of the four scales listed in table 2 are 1.23, .19, .12, and .09. The unweighted scale values unadjusted for a natural zero point afford the poorest values for satisfying the criterion of additivity. Scale values adjusted for their natural zero point by the addition-of-stimuli-procedure of Thurstone and Jones but without adjustment by stretching factors show marked improvement in precision. The precision is sharpened by use of stretching factors.

The improvement in precision is greater if scale values are transformed to maximize additivity. The F level for addition of the cubic term is 4.91. With 7 and 1 degrees of freedom, this falls between a 10% level of significance, 3.59, and a 5% level of significance, 5.59.

Further Comments

A theoretical point at issue is whether a scale metric based upon the standard deviation of the discriminal dispersion is sufficient to schieve additivity of scale values after correction for the natural zero point and adjustments for comparing single-single, single-double and double-double stimuli have been made. The analysis given here indicates that improved correction is obtained by a continuous functional transformation to maximize additivity.

The conclusion that the raw discriminal dispersion does not produce scale values whose proportionality after zero point correction assures additivity of scale values is not unexpected. In scaling sensory intensities rather than intensities of desire, the addition of scale values does not conform to the addition of physical stimulus intensities together. Instead, equal increments in physical stimulus produce diminishing increments in the sensory stimulus experienced.

Consideration of consumer purchasing indicates a similar result. The standard error in choosing between two similar articles worth $101 and $102 is much larger than in choosing between articles worth $1 and $2 to the individual judge. A previous paper by the author and John H. Platten, Jr., indicated that scale values based upon the discriminal dispersion of preference judgments require a Fechner-type correction in order for the relationship of additivity to be satisfied [1].

References

[1] Benson, Purnell H., and Platten, John H., Jr., "Preference Measurement by the Methods of Successive Intervals and Monetary Estimates," Journal of Applied Psychology, Vol. 40 (1956), pp. 412-414.

[2] Thurstone, Louis L. and Jones, Lyle V., "The Rational Origin for Measuring Subjective Values," Journal American Statistical Association, Vol. 52 (1957), pp. 458-471.